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An Improved k - ϵ Model for Near Wall Turbulence

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Abstract

This paper presents an improved k - ϵ model for low Reynolds number turbulence near a wall. The work is twofold: In the first part, the near-wall asymptotic behavior of the eddy viscosity and the pressure transport term in the turbulent kinetic energy equation are analyzed. Based on these analyses, a modified eddy viscosity model with the correct near-wall behavior is suggested, and a model for the pressure transport term in the k -equation is proposed. In addition, a modeled dissipation rate equation is reformulated, and a boundary condition for the dissipation rate is suggested. In the second part of the work, one of the deficiencies of the existing k - ϵ models, namely, the wall distance (e.g., y^+) dependency of the equations and the damping functions, is examined. An improved model that does not depend on any wall distance is introduced. Fully developed turbulent channel flows and turbulent boundary layers over a flat plate are studied as validations for the proposed new models. Numerical results obtained from the present and other previous k - ϵ models are compared with data from direct numerical simulation. The results show that the present k - ϵ model, with added robustness, performs as well as or better than other existing models in predicting the behavior of near-wall turbulence.

1. Introduction

The k - ϵ model is one of the most widely used turbulence models in engineering applications. Patel et al.^[1] recently reviewed existing two-equation models that can be integrated directly to the wall. One of their conclusions was that the damping functions used in turbulence models, especially the one for the eddy viscosity, need to be further modified in order to improve model performance. In fact, as we shall see later, many existing k - ϵ models do not provide the correct near-wall behavior of the eddy viscosity.

Shih^[2] recently proposed a new near-wall k - ϵ model based on asymptotic analysis. The present paper is a direct extension of that work.

In the present paper, we will first analyze, in section 2, the near-wall asymptotic behavior of the eddy viscosity and the pressure transport term in the k -equation, and in sections 3 and 4, propose models according to their near-wall behaviors. The model equation for the dissipation rate is reformulated following an argument similar to that of Lumley,^[5] and a boundary condition for ϵ is suggested.

An asymptotic analysis shows that, in the near wall region, while the pressure transport term in the turbulent kinetic energy equation is small compared to the dissipation and molecular diffusion terms, it is much larger than the turbulent transport term, and

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it plays an important role in the balance between the dissipation and molecular diffusion terms. This near-wall behavior is also observed in direct numerical simulation of fully developed channel flows (Mansour et al.^[32], Kim et al.^[4]). However, in existing $k-\epsilon$ models, this pressure transport term is either ignored or lumped into the turbulent transport model. The present work introduces a model for the pressure transport term explicitly.

Most of the existing $k-\epsilon$ models for near-wall turbulence use y^+ (defined as $u_\tau y/\nu$, where u_τ the friction velocity) as a parameter in constructing damping functions, with the Jones-Launder model being the only exception. While the use of y^+ is perfectly fine for simple attached boundary layer flows, it is inconvenient in more general applications such as separated flows and flows with corners, where y^+ is not well defined. The Jones-Launder model has the advantage of avoiding y^+ ; however, the model is known to perform poorly in predicting near wall turbulent quantities, especially the turbulence kinetic energy. In the present work, a new damping function is derived based on asymptotic analysis (Section 5). The new function is constructed upon a non-dimensional quantity that is independent of the coordinate system.

The new models proposed in this paper were validated using direct numerical simulation data for fully developed turbulent channel flows and turbulent boundary layers over a flat plate. These numerical results are reported in Section 6. Comparisons are also made with other popular $k-\epsilon$ models implemented in the same computer code. The numerical results show that the present model, in general, performs better than the existing models while providing added robustness.

2. Asymptotic Analysis

To analyze the near-wall asymptotic behavior of the eddy viscosity and other turbulent quantities, we expand the fluctuating velocities and pressure in Taylor series about the wall distance as follows:

$$\begin{aligned} u_1 &= b_1 y + c_1 y^2 + d_1 y^3 + \dots \\ u_2 &= c_2 y^2 + d_2 y^3 + \dots \\ u_3 &= b_3 y + c_3 y^2 + d_3 y^3 + \dots \\ p &= a_p + b_p y + c_p y^2 + d_p y^3 + \dots \end{aligned} \tag{1}$$

where the coefficients a_p, b_1, c_2, \dots are functions of x, z and t . Using the continuity and momentum equations, Mansour et al.^[3] derived the following relations between the coefficients,

$$\begin{aligned} 2c_2 &= -(b_{1,1} + b_{3,3}) \\ a_{p,1} &= 2\nu c_1 \\ a_{p,3} &= 2\nu c_3 \end{aligned} \tag{2}$$

The eddy viscosity is usually defined as

$$-\langle u_i u_j \rangle = \nu_T (U_{i,j} + U_{j,i}) - \frac{2}{3} k \delta_{ij} \tag{3}$$

where $\langle \rangle$ stands for ensemble average and $k \equiv \langle u_i u_i \rangle / 2$ is the turbulent kinetic energy. For plane shear flows, we can write from Eq. (3)

$$\nu_T = \frac{-\langle uv \rangle}{\partial U / \partial y} \quad (4)$$

and using Eq. (1), we obtain the near-wall asymptotic behavior of the eddy viscosity:

$$\nu_T = \frac{-\langle b_1 c_2 \rangle}{\langle b_1 \rangle} y^3 + \frac{-\langle b_1 d_2 + c_1 c_2 \rangle + 2\langle b_1 c_2 \rangle \langle c_1 \rangle}{\langle b_1 \rangle} y^4 + O(y^5) \quad (5)$$

That is, near the wall ν_T is $O(y^3)$. A correct eddy viscosity model should have this near-wall behavior. We shall see later that many existing models do not have this near-wall behavior. For later use, let us examine also the near-wall asymptotic behavior of the turbulent kinetic energy k and its dissipation rate $\epsilon \equiv \nu \langle u_{i,j} u_{i,j} \rangle$. Using Eq. (1), we obtain the following relations for the k and ϵ :

$$k = \frac{\langle b_1^2 \rangle + \langle b_3^2 \rangle}{2} y^2 + (\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle) y^3 + O(y^4) \quad (6)$$

$$\frac{\epsilon}{\nu} = \langle b_1^2 \rangle + \langle b_3^2 \rangle + 4(\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle) y + O(y^2) \quad (7)$$

In addition, the pressure transport term in the k -equation, $\Pi \equiv -\frac{1}{\rho} \langle u_i p_{,i} \rangle$, becomes (using Eqs (1) and (2))

$$\Pi = -2\nu(\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle) y + O(y^2) \quad (8)$$

The turbulent transport term in the k -equation, $-\langle k u_i \rangle_{,i}$, can be estimated as $O(y^3)$. Therefore, the pressure transport term is much larger than the turbulent transport term near the wall.

3. Eddy viscosity model

In this section, we will propose a model for the eddy viscosity using its near-wall behavior described in the previous section. In general, the eddy viscosity model can be written as

$$\nu_T = c u' \ell' \quad (9)$$

where u' and ℓ' are the turbulent characteristic velocity and length scale, respectively. Depending on how the velocity and length scales are specified, the eddy viscosity model can be a mixing length model, a one-equation (k) model or a two-equation (e.g. k - ϵ) model. For example, in plane shear flows, Prandtl's mixing length model specifies the characteristic velocity with $\ell' \partial U / \partial y$. For near wall turbulence, the Van Driest mixing length model further damps the length scale to $y[1 - \exp(-y^+/A)]$. For more advanced mixing length models, see Baldwin and Lomax^[6], and King^[7]. One-equation (k) models use $k^{1/2}$ as the characteristic velocity, which is determined by the turbulent kinetic energy

equation. In two-equation models, e.g. k - ϵ models, the length scale is usually specified by $k^{3/2}/\epsilon$, where ϵ is determined by a dissipation rate equation. In this paper we will concentrate on two-equation models, wherein the eddy viscosity is usually written as

$$\nu_T = C_\mu f_\mu \frac{k^2}{\epsilon} \quad (10)$$

where $C_\mu = 0.09$, and f_μ is a damping function. The form of the damping function is critical in such formulations, since the prediction of the mean velocity field depends primarily on the eddy viscosity model. Thus it is important for an eddy viscosity model to have the proper near-wall behavior. We have examined the near-wall behavior of eddy viscosity models based on various k - ϵ model equations. The results are listed in Table 1, which shows that some of the k - ϵ models do not have the correct near-wall behavior of the eddy viscosity, namely, $\nu_t = O(y^3)$.

The quantity $k^{3/2}/\epsilon$ is usually considered a characteristic length scale, ℓ' , (or the size) of the energy containing eddies. One expects that near the wall the size of these eddies to be of the order of the distance from the wall, $O(y)$. However, Eq.s (6) and (7) show that $k^{3/2}/\epsilon$ is $O(y^3)$. Hence, $k^{3/2}/\epsilon$ is not an appropriate quantity to represent the length scale of the large eddies near the wall. We therefore introduce a new variable $\tilde{\epsilon}$:

$$\tilde{\epsilon} = \epsilon - \nu \frac{\partial k / \partial x_i}{2k} \quad (11)$$

which has the following property: $\tilde{\epsilon}$ approaches ϵ away from the wall and is $O(y^2)$ near the wall. Therefore, $k^{3/2}/\tilde{\epsilon}$ is a proper quantity to characterize the length scale of the large eddies. With this length scale, the eddy viscosity should be written as

$$\nu_T = C_\mu f_\mu \frac{k^2}{\tilde{\epsilon}} \quad (12)$$

Now in order for ν_T to have the correct near-wall behavior, the damping function f_μ must be $O(y)$ near the wall and approaches 1 away from the wall. The damping functions used in various k - ϵ models are listed in Table 2. If we consider the presence of the wall as the main effect on the eddy viscosity, then we may assume f_μ is mainly a function of y^+ . The form of f_μ can be determined quite accurately if we know ν_T , k and $\tilde{\epsilon}$ from, for example, the direct numerical simulation. One may optimize the following simple form by numerical experiments:

$$f_\mu = 1 - \exp(-a_1 y^+ - a_2 y^{+2} - a_3 y^{+3} - a_4 y^{+4}) \quad (13)$$

The optimal values for channel flows are $a_1 = 6 \times 10^{-3}$, $a_2 = 4 \times 10^{-4}$, $a_3 = -2.5 \times 10^{-6}$, $a_4 = 4 \times 10^{-9}$. It can be shown that this form of damping function does provide the required near-wall behavior. As will be shown later, the above constants are valid for general boundary layer flows.

4. Modeled k - ϵ equation

To complete the eddy viscosity model, we need the modeled equations for the turbulent kinetic energy and its dissipation rate. In this section we will analyze the near-wall behavior of the k -equation and propose a model for the pressure transport term with a proper near-wall behavior. The equation for the dissipation rate is also reformulated with a formal invariant analysis.

4.1 k -equation

We start with the equation for the turbulent kinetic energy,

$$k_{,t} + U_j k_{,j} = D_\nu + T + \Pi + P - \epsilon \quad (14)$$

where D_ν , T and Π represent the transport of the turbulent kinetic energy due to the viscosity, turbulent velocity and pressure, respectively. P and ϵ are the rate of production and dissipation of the turbulent kinetic energy. The terms on the right hand side of Eq. (14) are defined as follows:

$$\begin{aligned} D_\nu &= \nu k_{,jj} \\ T &= -\langle k u_j \rangle_{,j} \\ \Pi &= -\frac{1}{\rho} \langle p u_j \rangle_{,j} \\ P &= -\langle u_i u_j \rangle U_{i,j} \\ \epsilon &\equiv \nu \langle u_{i,j} u_{i,j} \rangle \end{aligned} \quad (15)$$

Using Eq.s (1) and (2), we obtain the budget of the k -equation near the wall,

$$\begin{aligned} \frac{Dk}{Dt} &= O(y^3) \\ D_\nu &= \nu(\langle b_1^2 \rangle + \langle b_3^2 \rangle) + 6\nu(\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle)y + O(y^2) \\ T &= O(y^3) \\ \Pi &= -2\nu(\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle)y + O(y^2) \\ P &= O(y^3) \\ \epsilon &= \nu(\langle b_1^2 \rangle + \langle b_3^2 \rangle) + 4\nu(\langle b_1 c_1 \rangle + \langle b_3 c_3 \rangle)y + O(y^2) \end{aligned} \quad (16)$$

This budget shows that the term Π is much larger than the term T , and that Π cannot be neglected if we want the k -equation be balanced in the near-wall region. However, the existing models either do not consider this term or simply combine it with the term T and model them as

$$-\langle k u_j \rangle_{,j} = \left[\frac{\nu_T}{\sigma_k} k_{,j} \right]_{,j} \quad (17)$$

In this paper, we propose a model for Π which has a similar form to that of the standard turbulent transport model, but with a coefficient to ensure its correct near-wall behavior, Eq. (8). The proposed model form of Π is

$$\Pi = \left\{ \frac{C_0}{f_\mu [1 - \exp(-y^+)]} \frac{\nu_T}{\sigma_k} k_{,j} \right\}_{,j} \quad (18)$$

where $C_0 = 0.05$ is a model constant. In some existing k - ϵ models, it is assumed that $\epsilon = 0$ at the wall. In that case, in order to balance the term D_ν , a nonzero artificial term D must be added to the k -equation. The form of D for various k - ϵ models is listed in Table 3. Finally, the modeled k -equation becomes

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_k} \right) \frac{\partial k}{\partial x_j} \right] + \Pi + \nu_T \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) \frac{\partial U_i}{\partial x_j} - \epsilon + D \quad (19)$$

In the present model, $D = 0$, since ϵ is nonzero at the wall. (The boundary condition for ϵ will be discussed later.)

4.2 ϵ -equation.

The exact dissipation rate equation is

$$\epsilon_{,t} + U_j \epsilon_{,j} = D_\nu^\epsilon + T^\epsilon + \Pi^\epsilon + PD \quad (20)$$

where D_ν^ϵ , T^ϵ and Π^ϵ represent the diffusion rate of the dissipation rate due to the viscosity, turbulent velocity and pressure, respectively, and PD stands for the entire mechanism of the **production and destruction** of the dissipation rate ϵ . The terms on the right hand side of the above equation are as follows:

$$\begin{aligned} D_\nu^\epsilon &= \nu \epsilon_{,jj} \\ T^\epsilon &= -\nu \langle u_{i,k} u_{i,k} u_{j,j} \rangle_{,j} \\ \Pi^\epsilon &= -\frac{2\nu}{\rho} \langle p_{,k} u_{j,k} \rangle_{,j} \\ PD &= -2\nu (\langle u_{i,k} u_{j,k} \rangle + \langle u_{k,i} u_{k,j} \rangle) U_{i,j} - 2\nu \langle u_j u_{i,k} \rangle U_{i,kj} \\ &\quad - 2\nu \langle u_{i,k} u_{j,k} u_{i,j} \rangle - 2\nu^2 \langle u_{i,kj} u_{i,kj} \rangle \end{aligned} \quad (21)$$

The term Π^ϵ is usually neglected and the term T^ϵ is modeled as

$$T^\epsilon = \left[\frac{\nu_T}{\sigma_\epsilon} \epsilon_{,j} \right]_{,j} \quad (22)$$

To model PD , we define Ψ by

$$PD = -\frac{\epsilon \tilde{\epsilon}}{k} \Psi$$

At the level of the k - ϵ model, we assume Ψ is a function of ν , ν_T , k , ϵ , $\tilde{\epsilon}$, $U_{i,j}$ and $U_{i,jk}$. Since Ψ is an invariant, it must be a function of the invariants that can be constructed from these quantities. Therefore we can write

$$\Psi = \Psi(R_t, \frac{\nu_T U_{i,j} U_{i,j}}{\tilde{\epsilon}}, \nu \nu_T U_{i,jk} U_{i,jk} \frac{k}{\epsilon \tilde{\epsilon}})$$

where R_t is the turbulent Reynolds number $k^2/\nu\epsilon$. We expand Ψ in a Taylor series about $\nu_T U_{i,j} U_{i,j}/\tilde{\epsilon}$ and $\nu \nu_T U_{i,jk} U_{i,jk} k/\epsilon \tilde{\epsilon}$, and take only the linear terms. We obtain

$$\Psi = \psi_0 + \psi_1 \frac{\nu_T U_{i,j} U_{i,j}}{\tilde{\epsilon}} + \psi_2 \nu \nu_T U_{i,jk} U_{i,jk} \frac{k}{\epsilon \tilde{\epsilon}} \quad (23)$$

where the coefficients ψ_0 , ψ_1 and ψ_2 are in general functions of R_t . Finally, the modeled dissipation rate equation becomes

$$\frac{\partial \epsilon}{\partial t} + U_j \frac{\partial \epsilon}{\partial x_j} = \frac{\partial}{\partial x_j} \left[\left(\nu + \frac{\nu_T}{\sigma_\epsilon} \right) \frac{\partial \epsilon}{\partial x_j} \right] + C_1 f_1 \frac{\epsilon}{k} \nu_T U_{i,j} U_{i,j} - C_2 f_2 \frac{\epsilon \tilde{\epsilon}}{k} + E \quad (24)$$

where, C_1 and C_2 are the model constants, and f_1 and f_2 are functions of R_t . The term E in the present model comes from the last term in Eq. (23):

$$E = \nu \nu_T U_{i,j,k} U_{i,j,k} \quad (25)$$

where we have taken $\psi_2 = -1$. The form of E and C_1 , C_2 , f_1 and f_2 for various k - ϵ models are listed in Tables 3 and 4.

4.3 Boundary Condition for ϵ .

Many of the earlier k - ϵ models use $\epsilon = 0$ as the boundary condition for the dissipation rate. It is now generally agreed that this is not the physically correct boundary condition. However, controversy still exists in what boundary condition should be used for the dissipation rate. In some calculations, $\partial \epsilon / \partial y = 0$ is used, which clearly has no physical background. Most models use the second derivative of the turbulent kinetic energy at the boundary as the boundary condition for ϵ , as listed in Table 1. This condition comes directly from the k -equation and is physically correct; however, it makes the problem very stiff and thus put very stringent restrictions on the choice of initial profiles for k and ϵ . If the initial profile for k is not given correctly, the second derivative of k can become negative and cause the solution procedure to diverge.

We propose the following boundary condition based on the asymptotic analysis. At $y = 0$, it is obvious from eqs. (6) and (7) that

$$\epsilon = 2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2 \quad (26)$$

This expression is exact at the wall, and it does not add stiffness to the solution procedure.

5. Deficiencies and Improvements of Existing Models

5.1 Damping functions

One of the deficiencies of the existing near wall models is that most of the wall damping coefficients are functions of a wall coordinate, such as the y^+ . This types of damping function works well only in the cases of attached boundary layer flows with simple geometries where y^+ is well defined. For practical engineering problems with corner flows or separated flows, some ad hoc treatment to the damping function must be made. The same is also true for the length scale in an algebraic model. The only exception to the above is the Jones-Launder model in which the damping function is a function of $R_t = k^2 / \nu \epsilon$.

Although the Jones-Launder model has the advantage of independent of y^+ , it is known that this model does not predict correctly the near-all turbulence, especially,

the near-wall turbulent kinetic energy is underpredicted. One natural option could be to modify the J-L model such that it would predict the correct near-wall turbulent quantities. However, one basic difficulty met in such attempts is that the k -equation is very stiff. The fact that we are putting k^2 back into the equation by using the parameter R_t aggravates the situation.

To avoid the above difficulty and yet still achieve the same end, we introduce the following parameter:

$$R_u = \frac{U^4}{\nu \epsilon}, \quad (27)$$

where U is the total velocity. (Note: U is the total velocity in a coordinate frame fixed to the solid boundary.) From the results given in Section 3, since ϵ approaches a finite value at the wall, it is obvious from eq. (1) that $R_u = O(y^4)$ near the wall. Similar to eq. (13) of Section 3, we write the damping function for the eddy viscosity as:

$$f_\mu = 1 - \exp(-a_1 R_u^{1/4} - a_2 R_u^{1/2} - a_3 R_u) \quad (28)$$

with $a_1 = 5 \times 10^{-3}$, $a_2 = 7 \times 10^{-5}$, and $a_3 = 8 \times 10^{-7}$. One can easily verify that with this damping function, the eddy viscosity has the correct near wall behavior, i.e., $\nu_T = O(y^3)$.

One point worth mentioning is the wide applicability of the above damping function. Though developed with Shih model (Section 3, 4) in mind, it can be used with any existing $k - \epsilon$ model that uses a non-zero boundary condition for the dissipation. This new parameter R_u , unlike R_t , by no means affect the stability of the solution procedure.

5.2 Pressure transport term.

In order to remove the coordinate dependency of the k -equation, we replace the pressure transport term given in eq. (18) by the following expression:

$$\Pi = \left[\frac{C_0}{f_\mu^2} \frac{\nu_T}{\sigma_k} k_{,j} \right]_{,j} \quad (29)$$

where the model constant is readjusted to $C_0 = 0.01$.

5.3 The formulation of $\tilde{\epsilon}$.

In order to obtain the correct wall behavior for the eddy viscosity, we have introduced $\tilde{\epsilon}$ in eq. (11), Section 3. Theoretical analysis shows that $\tilde{\epsilon}$ is always positive. However, in numerical calculations, the value of $\tilde{\epsilon}$ may become negative or even oscillatory due to round of errors. (Depending on the accuracy of the numerical procedure, this may or may not be the case.) Here, an alternative definition of $\tilde{\epsilon}$ is suggested:

$$\tilde{\epsilon} = (1 - \exp(-R_t^{1/2}))\epsilon \quad (30)$$

This expression has the same near wall behavior as eq. (11) but is less likely to cause numerical instability.

With the modifications suggested above, the model constants need slight adjustment. The constants used in the present modified model are as shown in Table 4.

6. Numerical Testing

Flows with self-similar solutions are particularly useful for accurate model testing, because their solutions are independent of initial conditions, and one does not need to choose carefully the initial conditions for the k and ϵ . In this paper, we use a fully developed channel flow and a flat plate boundary layer for model testing. These flows are the simplest wall bounded turbulent shear flows with self-similar solutions. However, the complex features of the turbulence, for example, the effect of the wall on shear turbulence, are present. In the case of the channel flow, the k - ϵ equations form a one-dimensional problem, numerical calculations are easy and accurate. Recently, the measurements^[8] confirmed the accuracy of the direct numerical simulation data.^[4] These data are used for model validations.

In legends of the figures presented at the end of the paper, which will be discussed in detail in the following paragraphs, the word “present” refers to results obtained using the model suggested in Section 4 together with the new damping function and $\tilde{\epsilon}$ given in Section 5, while the label “Shih” refers to results obtained using strictly formulations given in Section 4.

6.1 Fully developed turbulent channel flow

Let h be the half width of the channel, u_τ the friction velocity and Re_τ the Reynolds number defined as $u_\tau h/\nu$. Let U, k, ϵ, ν_T and y be the non-dimensional quantities, normalized by $u_\tau, u_\tau^2, u_\tau^3/h, \nu$ and h , respectively. The modeled equations for the channel flow become

$$\frac{dU}{dy} = Re_\tau \frac{1-y}{1+\nu} \quad (31)$$

$$\frac{d}{dy} \left\{ \frac{1}{Re_\tau} \left[1 + (1+C) \frac{\nu_T}{\sigma_k} \right] \frac{dk}{dy} \right\} + \nu_T \left(\frac{dU}{dy} \right)^2 \frac{1}{Re_\tau} - \epsilon = 0 \quad (32)$$

$$\frac{d}{dy} \left\{ \frac{1}{Re_\tau} \left(1 + \frac{\nu_T}{\sigma_\epsilon} \right) \frac{d\epsilon}{dy} \right\} + C_1 \frac{\epsilon}{k} \nu_T \left(\frac{dU}{dy} \right)^2 \frac{1}{Re_\tau} - C_2 f_2 \frac{\epsilon \tilde{\epsilon}}{k} + \nu_T \left(\frac{d^2 U}{dy^2} \right)^2 \frac{1}{Re_\tau^2} = 0 \quad (33)$$

where

$$\begin{aligned} \nu_T &= C_\mu f_\mu Re_\tau \frac{k^2}{\tilde{\epsilon}} \\ \tilde{\epsilon} &= \epsilon - \frac{(\frac{dk}{dy})^2}{2k Re_\tau} \\ f_\mu &= \text{equation(13)} \\ f_2 &= 1 - \frac{0.4}{1.8} \exp\left[-\left(\frac{Re_\tau k^2}{6\epsilon}\right)^2\right] \\ C &= \frac{C_0}{f_\mu [1 - \exp(-y^+)]} \end{aligned} \quad (34)$$

The boundary conditions are simple. At the wall,

$$\begin{aligned} U &= k = 0 \\ \epsilon &= \frac{\left(\frac{dk}{dy}\right)^2}{2k Re_\tau} \end{aligned} \quad (35)$$

and at the center of the channel,

$$\frac{dk}{dy} = \frac{d\epsilon}{dy} = 0 \quad (36)$$

The main results from different k - ϵ models for a channel flow with $Re_\tau = 180$ are plotted in figures 1 – 4. All the calculations are compared with the direct numerical simulation data. Figure 1 shows mean velocity profile, Figure 2 shows the turbulent kinetic energy, Figure 3 shows the turbulent shear stress distribution, and Figure 4 shows the Dissipation rate. From these numerical results we reach the following conclusions: The model of Jones and Launder^[10] (JL) underpredicts the mean velocity as well as the peak value of the turbulent kinetic energy. Chien's model^[11] performs better than the JL model, but it overpredicts the mean velocity near the center of the channel as well as the turbulent kinetic energy. In these two models, $\epsilon = 0$ at the wall is used as the boundary condition, so the dissipation rate near the wall cannot be correctly predicted. Lam and Bremhorst^[12] use a nonzero boundary condition for ϵ and have made some improvement for the mean velocity and turbulent kinetic energy compared with the results of the JL model. However, the shear is much overpredicted, and the dissipation rate near the wall is not correct. The model of Nagano and Hishida^[13] presents a very good prediction for the mean velocity and shear stress, while the peak value of k is underpredicted. Their main modification to the JL model is a change in the damping function f_μ and the form of E . A zero dissipation rate at the wall is used. The numerical results from the Shih model and the present modified Shih model show improvements in the prediction of all quantities, including the dissipation rate.

6.2 Boundary layer flows

The boundary layer equations and the corresponding k - and ϵ -equations are solved using a conventional semi-implicit finite difference scheme. In this scheme, the coefficients for the convection terms are lagged one step in the x -direction, and the source terms in the k - and ϵ -equation are linearized in such a way that stability is ensured.

In the present study, a grid of 100 points in the y -direction is used. The grid is stretched linearly with $\Delta y_{j+1}/\Delta y_j = 1.05$. The grids expands in the y -direction according to the boundary layer growth rate.

The results of this calculation are presented in Figure 5 through Figure 10.

Figure 5 and 6 show the comparison of the wall shear stress from various models to experimental data and some DNS data. As shown in Figure 5, at low Reynolds number (based on momentum thickness), J-L model overpredicts the shear stress while Chien model and NH model underpredict the shear stress. One common character of these three models is that they all used zero boundary condition for the dissipation, and thus unable

to predict the correct turbulence near wall behavior. At high Reynolds number (Figure 6) the JL model still overpredicts the wall shear stress, while the others seem to do a fare job.

Figure 7 and 8 are the results for $Re_\theta = 1410$, and Figure 9 and 10 are the results for $Re_\theta = 7700$. From these results one can reach similar conclusions as we did from the channel flow case: The JL model generally underpredicts the peak value of the turbulent kinetic energy while overpredicts it in the inertia layer; the model also underpredicts the mean velocity profiles. The LB model and the NH model also underpredict the peak value of the turbulent kinetic energy near the wall. The three models mentioned above either have zero boundary condition for ϵ or do not have the correct order of magnitude for eddy viscosity. The Figures show that, in general, the present model performs better than the existing models.

Conclusions

From the model testing, we conclude that the present $k-\epsilon$ model has made considerable improvement over previous $k-\epsilon$ models according to the comparison with the direct numerical simulation data. We find that the improvement is mainly due to the modified eddy viscosity model and the model of the pressure transport term in the k -equation. The proposed dissipation rate equation also shows a better near-wall behavior than the previous ones. The correct boundary condition for ϵ also seem to play an important role in the accurate prediction of the turbulence near wall behavior.

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Table 1 Eddy viscosity and boundary condition for ϵ in various k - ϵ models

Model	ν_T	BC: ϵ_w
JL	$O(y^3)$	0
Reynolds	$O(y^5)$	$\nu \frac{\partial^2 k}{\partial y^2}$
LB	$O(y^4)$	$\nu \frac{\partial^2 k}{\partial y^2}$
Chien	$O(y^3)$	0
NH	$O(y^4)$	0
Shih	$O(y^3)$	$2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$
Present	$O(y^3)$	$2\nu \left(\frac{\partial \sqrt{k}}{\partial y} \right)^2$

Table 2 Damping functions used in various k - ϵ models

Model	f_μ	f_1	f_2
JL	$\exp(\frac{-2.5}{1+R_t/50})$	1.0	$1 - .3 \exp(-R_t^2)$
Reynolds	$1 - \exp(-.0198R_k)$	1.0	$[1 - .3 \exp(-R_t^2/9)] f_\mu$
LB	$[1 - \exp(-.0165R_k)]^2$	$1 + (\frac{.05}{f_\mu})^3$	$1 - \exp(-R_t^2)$
	$\times (1 + \frac{20.5}{R_t})$		
Chien	$1 - \exp(-.0115y^+)$	1.0	$1 - .22 \exp(-R_t^2/36)$
NH	$[1 - \exp(-y^+/26.5)]^2$	1.0	$1 - .3 \exp(-R_t^2)$
Shih	Eq. (13)	1.0	$1 - .22 \exp(-R_t^2/36)$
Present	Eq. (28)	1.0	$1 - .22 \exp(-R_t^2/36)$

Table 3 Model terms in various k - ϵ models

Model	Π	D	E
JL	0	$-2\nu \left(\frac{\partial \sqrt{K}}{\partial y} \right)^2$	$2\nu \nu_T \left(\frac{\partial^2 U}{\partial y^2} \right)^2$
Reynolds	0	0	0
LB	0	0	0
Chien	0	$-\frac{2\nu K}{y^2}$	$-\frac{2\nu \epsilon}{y^2} \exp(-.5y^+)$
NH	0	$-2\nu \left(\frac{\partial \sqrt{K}}{\partial y} \right)^2$	$\nu \nu_T (1 - f_\mu) \left(\frac{\partial^2 U}{\partial y^2} \right)^2$
Shih	Eq. (18)	0	$\nu \nu_T \left(\frac{\partial^2 U}{\partial y^2} \right)^2$
Present	Eq. (29)	0	$\nu \nu_T \left(\frac{\partial^2 U}{\partial y^2} \right)^2$

Table 4 Model constants in various k - ϵ models

Model	C_μ	C_1	C_2	σ_k	σ_ϵ
JL	.09	1.45	2.0	1.0	1.3
Reynolds	.084	1.0	1.83	1.69	1.3
LB	.09	1.44	1.92	1.0	1.3
Chien	.09	1.35	1.8	1.0	1.3
NH	.09	1.45	1.9	1.0	1.3
Shih	.09	1.45	2.0	1.3	1.3
Present	.09	1.5	2.0	1.3	1.3

$$R_t = K^2/\nu\epsilon, R_k = \sqrt{K}y/\nu, y^+ = u_\tau y/\nu.$$

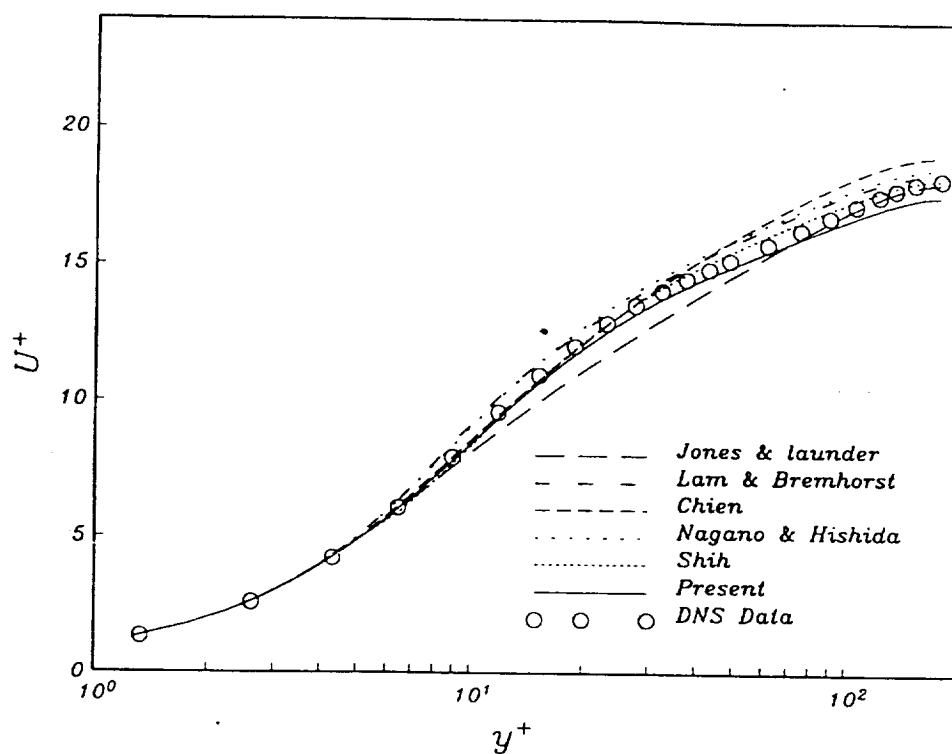


Figure 1. Mean velocity profile for channel flow, $Re_\tau = 180$.

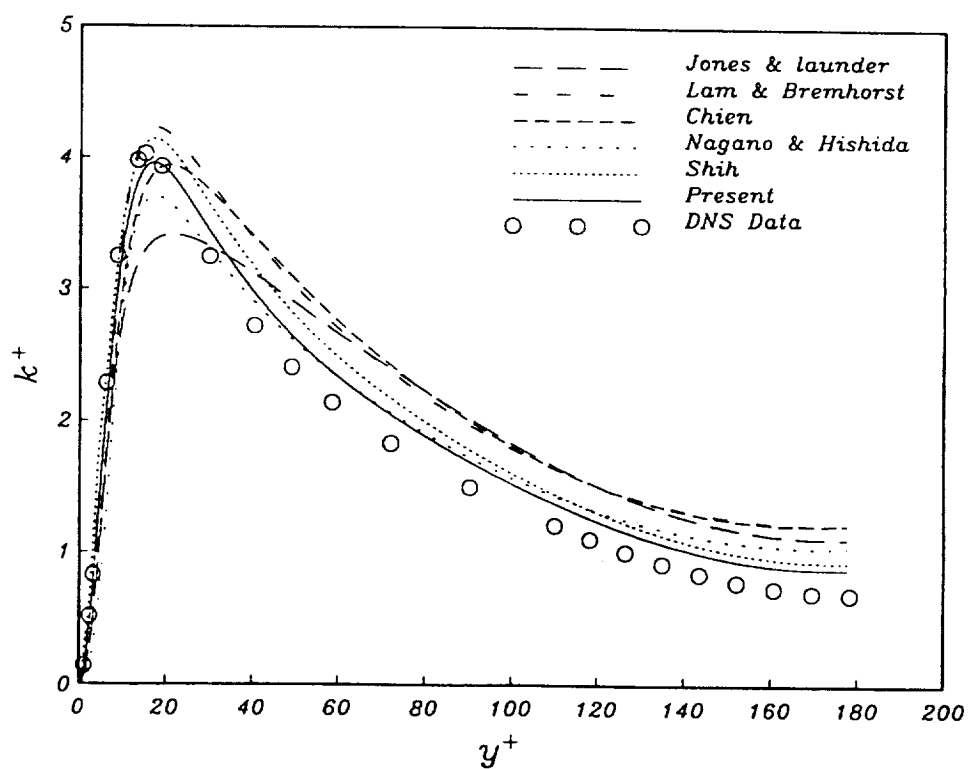


Figure 2. Turbulent kinetic energy for channel flow, $Re_\tau = 180$.

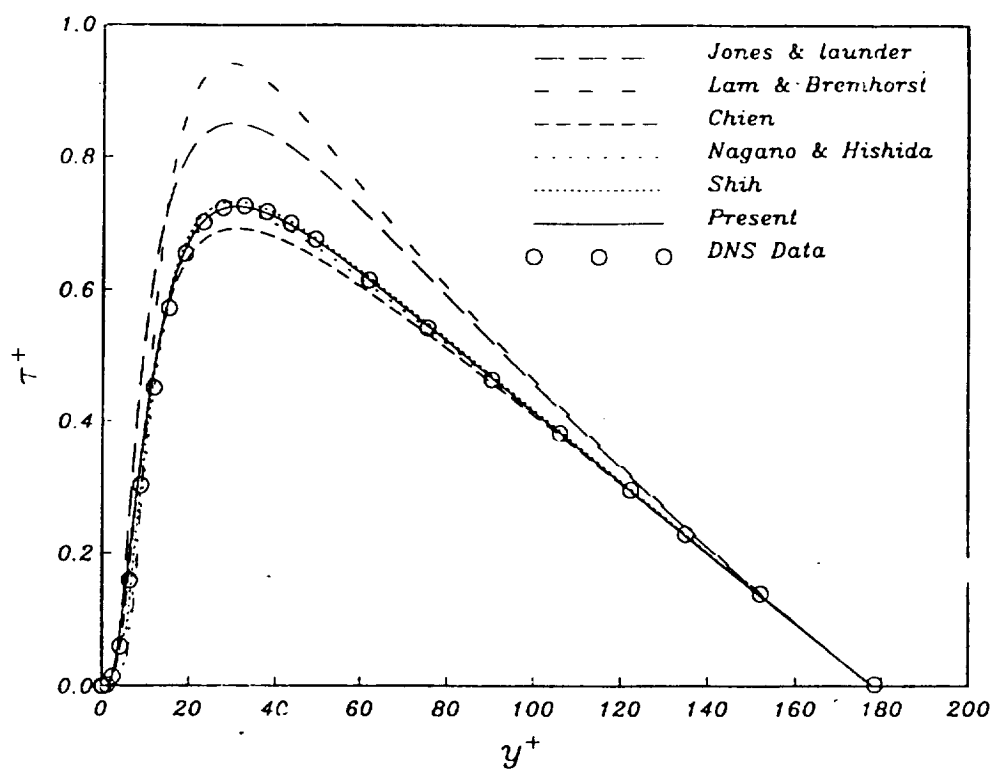


Figure 3. Turbulent shear stress for channel flow, $Re_\tau = 180$.

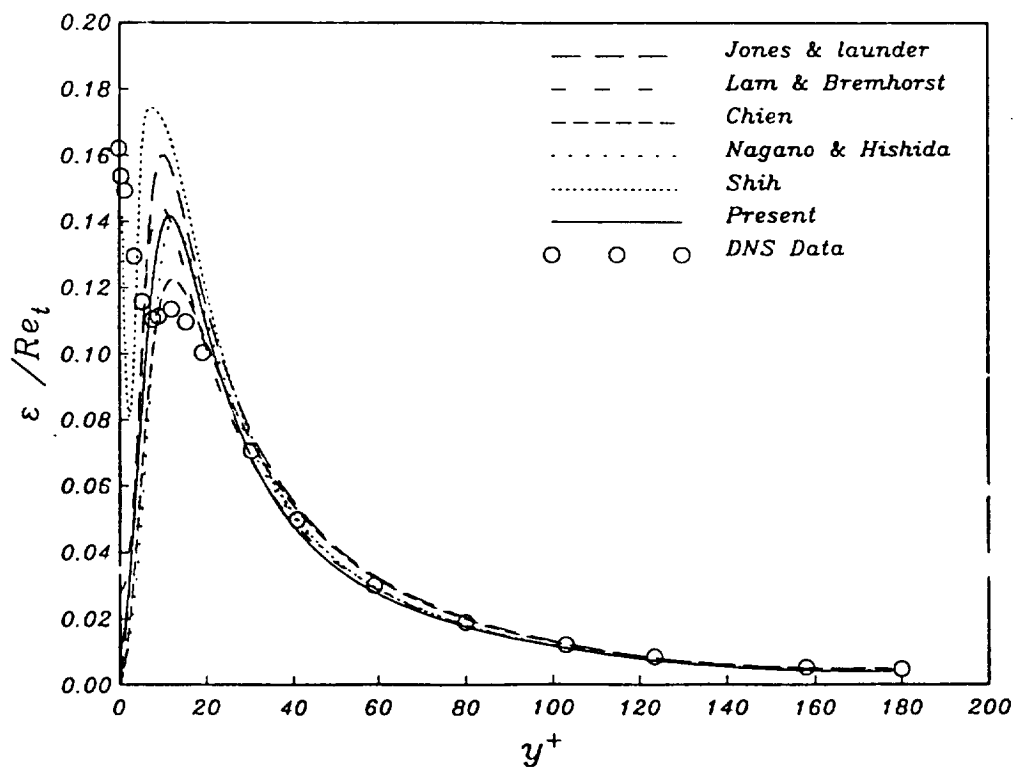


Figure 4. Dissipation rate for channel flow, $Re_\tau = 180$.

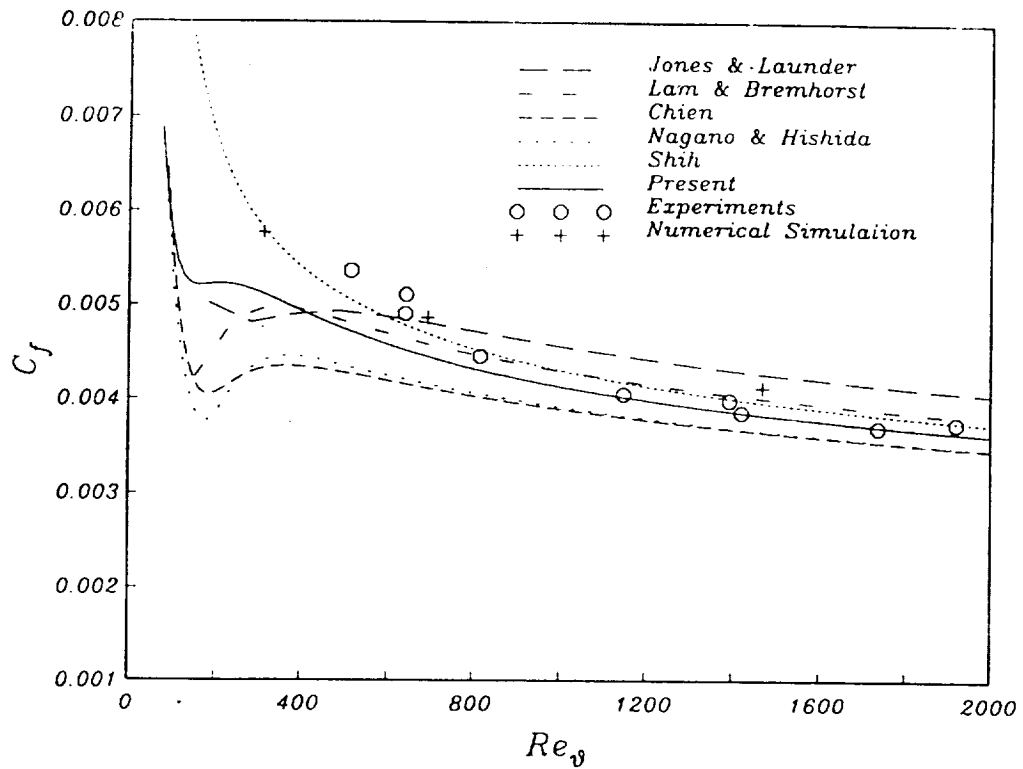


Figure 5. Wall shear stress coefficient, flat plate boundary layer flows, low Reynolds number data.

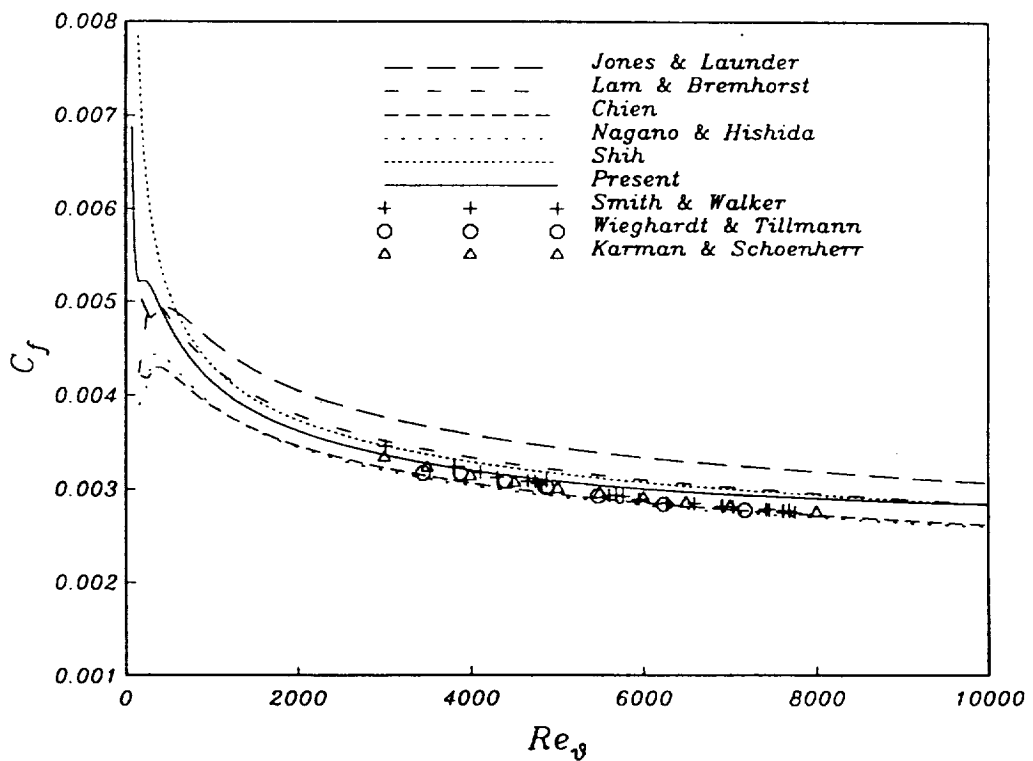


Figure 6. Wall shear stress coefficient, flat plate boundary layer flows, high Reynolds number data.

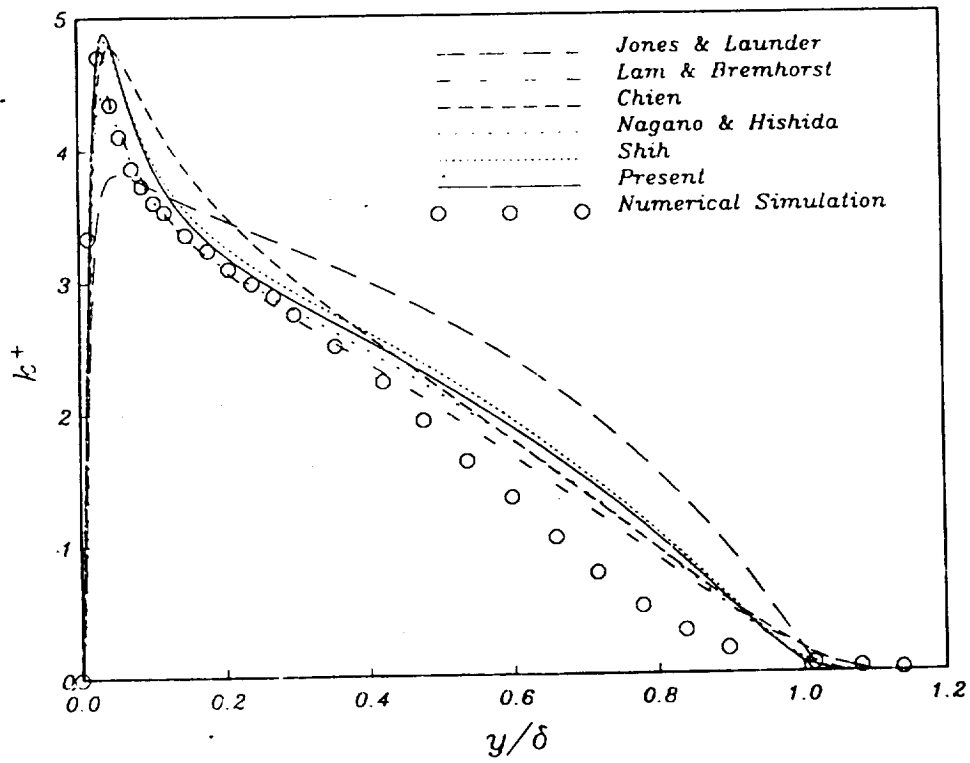


Figure 7. Turbulent kinetic energy for flat plate boundary layer, $Re_\theta = 1410$.

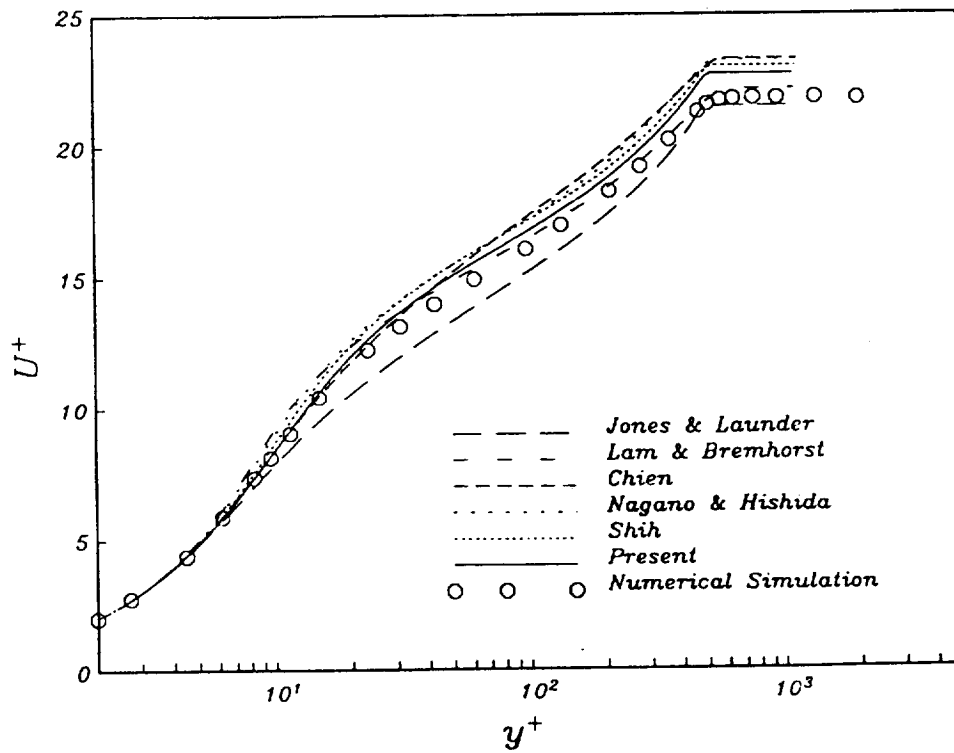


Figure 8. Mean velocity profile for flat plate boundary layer, $Re_\theta = 1410$.

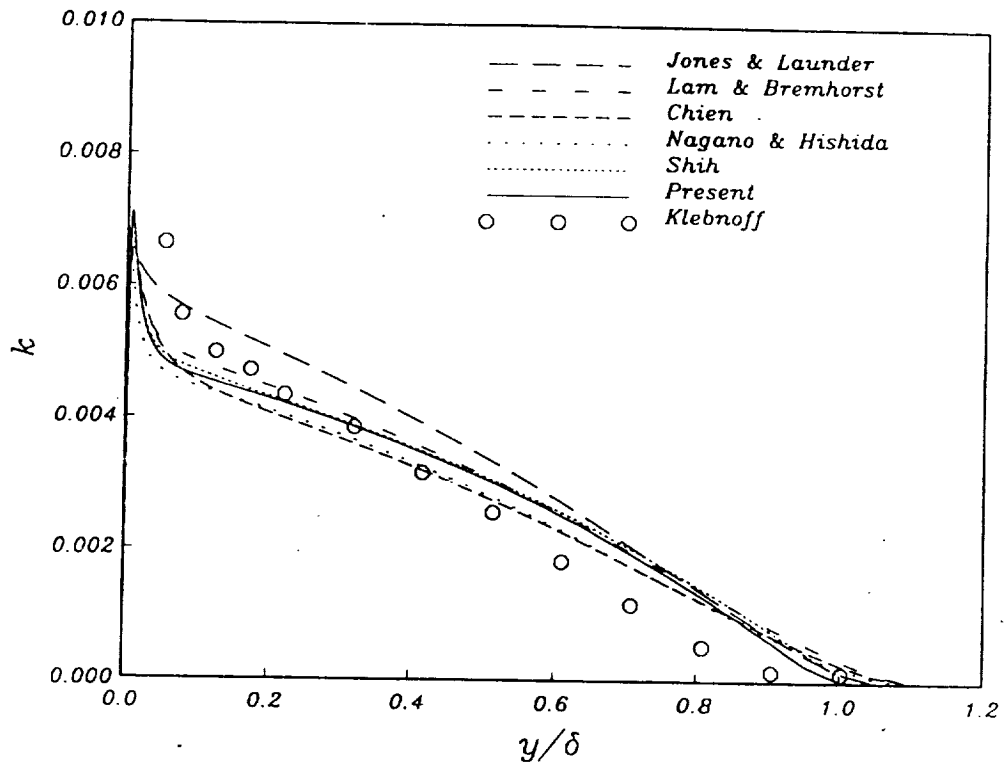


Figure 9. Turbulent kinetic energy for flat plate boundary layer, $Re_\theta = 7700$.

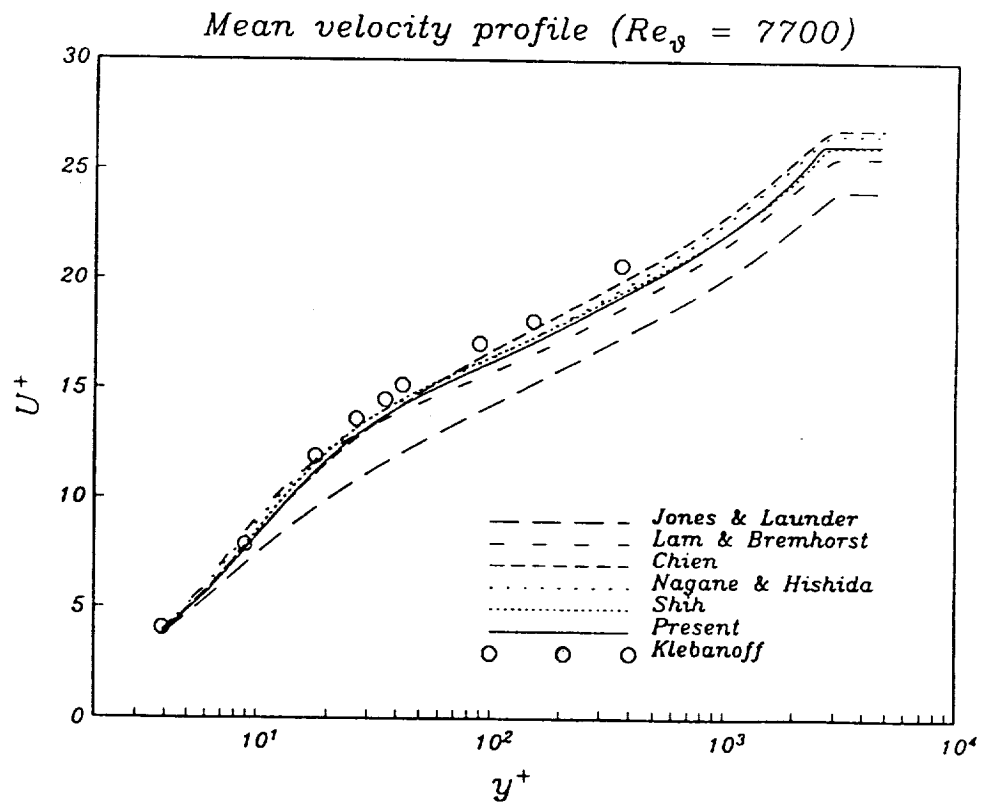


Figure 10. Mean velocity profile for flat plate boundary layer, $Re_\theta = 7700$.

SOURCE: NASA TM 104368, ICOMP-91-06, CMOTT-91-01

Low Reynolds Number Two-Equation Modeling of Turbulent Flows

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Abstract

A new $k - \epsilon$ turbulence model that accounts for viscous and wall effects is presented. The proposed formulation does not contain the local wall distance thereby making very simple the application to complex geometries. The formulation is based on an existing $k - \epsilon$ model that proved to fit very well with the results of direct numerical simulation. The new form is compared with nine different two-equation models and with direct numerical simulation for a fully developed channel flow at $Re = 3300$. The simple flow configuration allows a comparison free from numerical inaccuracies. The computed results prove that few of the considered forms exhibit a satisfactory agreement with the channel flow data. The new model shows an improvement with respect to the existing formulations.

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MODELING OF NEAR-WALL TURBULENCE

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ABSTRACT

This paper presents an improved $k-\epsilon$ model and a second order closure model for low-Reynolds number turbulence near a wall. For the $k-\epsilon$ model, a modified form of the eddy viscosity having correct asymptotic near-wall behavior is suggested, and a model for the pressure diffusion term in the turbulent kinetic energy equation is proposed. For the second order closure model, we modify the existing models for the Reynolds-stress equations to have proper near-wall behavior. A dissipation rate equation for the turbulent kinetic energy is also reformulated. The proposed models satisfy realizability and will not produce unphysical behavior. Fully developed channel flows are used for model testing. The calculations are compared with direct numerical simulations. It is shown that the present models, both the $k-\epsilon$ model and the second order closure model, perform well in predicting the behavior of the near wall turbulence. Significant improvements over previous models are obtained.

11-1125

SOURCE: NASA TM 104413, ICOMP-91-09, CMOTT-91-03

**ADVANCES IN MODELING
THE PRESSURE CORRELATION TERMS
IN THE SECOND MOMENT EQUATIONS**

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ABSTRACT

In developing turbulence models, different authors have proposed various model constraints in an attempt to make the model equations more general (or universal). The most recent of these are the realizability principle (Lumley 1978, Schumann 1977), the linearity principle (Pope 1983), the rapid distortion theory (Reynolds 1987) and the material indifference principle (Speziale 1983). In this paper we will discuss several issues concerning these principles and will pay special attention to the realizability principle raised by Lumley (1978). Realizability (defined as the requirement of non-negative energy and Schwarz' inequality between any fluctuating quantities) is the basic physical and mathematical principle that any modeled equation should obey. Hence, it is the most universal, important and also the minimal requirement for a model equation to prevent it from producing unphysical results. In this paper we will describe in detail the principle of realizability, derive the realizability conditions for various turbulence models, and propose the model forms for the pressure correlation terms in the second moment equations. Detailed comparisons of various turbulence models (Launder et al. 1975, Craft et al. 1989, Zeman and Lumley 1976, Shih and Lumley 1985 and one proposed here) with experiments and direct numerical simulations will be presented. As a special case of turbulence, we will also discuss the two-dimensional two-component turbulence modeling.